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Sequential tests for criticality of branching process with immigration*

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Abstract

We consider testing the criticality of the offspring mean against the local alternative including super- and sub- hypotheses, for branching process with immigration. Two sequential test procedures are discussed: a sequential criticality test (SCT) and a sequential probability ratio test (SPRT). For branching process with or without immigration, Nagai, Hitomi, Nishiyama and Tao (2020) proposed SCT. The SCT employs the observed Fisher information to define the stopping time, and is found to be a Z-test for local alternative including sub- and super- critical hypotheses. The SPRT minimized the average sample size under the null and alternative with no greater error probabilities. However, the testing result of SPRT is highly dependent on the prespecified the alternative probability densities. Simulation studies are conducted to verify our asymptotic results.

Keywords

Observed Fisher information; DDS Brownian motion; Bessel process; Joint Laplace transform; Z-test

1 Introduction

Branching processes have a number of applications in the physical and biological science, for example, cell kinetics, population growth, etc. Branching processes approximation can also be used to characterize the basic reproduction number in the early stage of epidemic processes. The behaviors of branching processes are quite different depending on whether the offspring mean is larger or smaller than 1. Due to great demand for

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statistical procedures to monitor the pandemic, we consider sequential tests for the criticality of Galton-Watson branching processes with immigration.

In sequential scheme, the observations can be obtained unceasingly and sequentially. We stop sampling when we get the "sufficient" information, then we do a statistical inference or make a statistical decision. Therefore, sequential sampling schemes are desirable for the quick detection of anomalies. A sequential method is characterized by two components: a stopping rule and a decision rule. The stopping rule decides whether to stop sampling with $(X_1, X_2, \ldots, X_n)$ or to continue to get an additional sample X_{n+1} for $n \geq 1$. The time when we stop sampling is called a stopping time. After the sampling stopped, the decision rule specifies the action for the problem of interest, for example, estimation, hypothesis testing, detection, etc.

We consider a branching process with immigration, let $\{Z_n\}$ be the nth generation size of a Bienaymé-Galton-Watson processes;

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k} + Y_n, \quad Z_0 > 0, n \in \mathbb{N}, \tag{1}$$

where $\xi_{n,k} \sim i.i.d. \left(m, \sigma^2\right)$ is the number of offspring of the ith individual belonging to nth generation and $Y_n \sim i.i.d. \left(\lambda, \sigma_Y^2\right)$ is the number of the immigration in the nth generation. The initial value Z_0 is a random variable which is independent of $\{\xi_{n,i}\}$.

We consider two type of stopping time and want to test the criticality of the offspring mean m in sequential schemes. One test is SCT, the stopping time is based on the observed Fisher information of the offspring mean m; for a predetermined c

$$\tau_c \equiv \inf \left\{ N > 1 : \sum_{n=1}^{N} Z_{n-1}/s_N^2 \geq c \right\}, \tag{2}$$

where s_N^2 is a natural estimator of σ^2written as

$$s_N^2 = \sum_{n=1}^{N} 1_{\{Z_{n-1}>0\}} \left(Z_n - Y_n - \hat{m}_N Z_{n-1}\right)^2 / (N Z_{n-1}).$$

In SCT, one could predetermine c, which guarantees the accuracy of the estimation. The stopping time τ_c in (2) based on the observed Fisher information is introduced by Lai and Siegmund (1983) [12] for the first order autoregressive model. They propose proceeding with the collection of the observations until to the observed Fisher information attains a predetermined level c and carrying out the fixed accuracy sequential estimation with the least square estimator. Using Lai and Siegmund's type stopping time, Sriram, Basawa, and Huggins (1991) [6] considered the fixed accuracy sequential estimation of offspring mean for branching process with immigration.

The sequential test based on the observed Fisher information, was considered by Nagai, Nishiyama and Hitomi (2018) [9], for testing unit root against local alternative under sequential sampling in a first order autoregressive processes. They proposed three unit root tests under the sequential sampling, using the sequential LSE, stopping time, and the combination of the two as the test statistics. They considered approximating discrete-time processes to continuous-time processes in space $D\,[0,\infty)$. Space $D\,[0,\infty)$ is the set of right-continuous functions on $[0,\infty)$ with left limits (See Billingsley (1999) for details). It is very natural to use $D[0,\infty)$ for characterizing the limiting behavior of sequential statistics, since in sequential analysis stopping times often take random values in unbounded time region. Hitomi, Nagai, Nishiyama and Tao (2019, 2020) [7, 5] obtained the joint density and joint Laplace transform of the sequential least square estimator and the stopping times. With the limiting joint Laplace transform and density function under the null and local alternatives, the operating characteristics of the tests can be computed from evaluations of some special functions such as parabolic cylinder functions and hypergeometric functions. Hitomi, Nagai, Nishiyama and Tao (2020)[6] also extended the sequential unit root test to a p-th order autoregressive models.

Nagai, Hitomi, Nishiyama and Tao (2020, 2021) [14, 8] considered the sequential testing problems on near-criticality in branching processes with or without immigration. We also obtained the joint density and Laplace transform of the test statistics and stopping time from the joint Laplace transform with respect to a Bessel process driven by Dambis-Dubins-Schwartz (DDS) Brownian motion. We found that the singularity of testing the near criticality in branching process is similar to that of testing the local-to-unity hypotheses in AR(1) process. For these two processes, using the stopping time based on the observed Fisher information, Nagai, Hitomi, Nishiyama and Tao (2020) [10] found that Bessel processes and Bessel bridges play important roles to analyze operating characteristics, such as size, powers, and moments in those sequential testing problems.

The other test is SPRT, with prespecified null hypothesis H_0 and alternative hypothesis H_1, the stopping time is defined as

$$T = \inf\{n > 1 : l_n \notin [a,b]\}\,. \tag{3}$$

where l_n is the log likelihood ratio between H_0 and H_1. SPRT accepts H_0 if $l_T < a$ and reject H_0 if $l_T > b$. a, b could be computed based on prescribed values of type I error probability α and type II error probability β.

As an efficient testing method of anti-aircraft gunnery, SPRT was developed by Wald (1945) [16]. In SPRT, besides the probability densities of H_0 and H_1, one need also to set the values of α and β in advance, so that such error probability could be guaranteed, when either hypotheses is in the case. It has been shown that SPRT minimizes the average sample size among sequential testing procedures under the null and alternative with no greater error probabilities. However, the testing result of SPRT is highly dependent on the prespecified the alternative probability densities.

This paper is organized as follows. Section 2 describes the models and testing hypotheses, and shows preliminary asymptotic results. In Section 3, we provide, without proof, the asymptotic properties of SCT, which is a part of Nagai, Hitomi, Nishiyama and Tao (2020, 2021) [14, 8]. Section 4 describes the SPRT. Section 5 reports the simulation results of the two tests. Section 6 concludes.

2 Model setting

We consider sequential test for the criticality of branching processes with immigration. Suppose $\{\xi_{n,k}\}_{n,k\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are independent, nonnegative, integer-valued random variables with $\left(m, \sigma^2\right)$ and $\left(\lambda, \sigma_Y^2\right)$ and independent each other. Let $\{Z_n\}$ be the nth generation size of a Galton-Watson processes;

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k} + Y_n, \quad Z_0 > 0, n \in \mathbb{N}, \tag{4}$$

where $\xi_{n,k}$ is the number of offspring of the ith individual belonging to nth generation and Y_n is the number of the immigration in the nth generation. The initial value Z_0 is a random variable which is independent of $\{\xi_{n,i}\}$.

We consider the probabilities with local parameters;

$$\text{Under} \quad P^0 : \left(m, \sigma^2\right) = \left(1, \sigma^2\right), \quad \text{Under} \quad P^\delta : \left(m, \sigma^2\right) = \left(1 + \delta/\sqrt{c}, \sigma_c^2\right),$$

with assumption $\sigma_N^2 \to \sigma^2$, as $c \to \infty$. The hypotheses for sub-critical test (right sided test) is

$$H_0 : \delta \geq 0 \quad \text{vs} \quad H_1 : \delta < 0, \tag{5}$$

and super-critical test is

$$H_0 : \delta \leq 0 \quad \text{vs} \quad H_1 : \delta > 0. \tag{6}$$

Suppose we have a sample (Z_n, Y_n), $n = 1, 2, \ldots, N$ from (4). When $\left\{\xi_k^{(n)}\right\}$ and $\{Y_n\}$ have power series distributions

$$P\left(\xi_k^{(n)} = j\right) = a_j \theta^j / A(\theta), \quad P\left(Y_n = k\right) = b_k \phi^k / B(\phi), \tag{7}$$

the maximum likelihood estimation $\hat{m}_N$ and the observed Fisher information of m are

$$\hat{m}_N = \sum_{n=1}^{N} (Z_n - Y_n) / \sum_{n=1}^{N} Z_{n-1}, \quad I_N(m) = \sum_{n=1}^{N} Z_{n-1}/\sigma^2. \tag{8}$$

The estimators of σ^2, λ, σ_Y^2 are

$$s_N^2 = \sum_{n=1}^{N} \frac{(Z_n - Y_n - \hat{m}_N Z_{n-1})^2}{Z_{n-1}} 1\{Z_{n-1} > 0\}, \tag{9}$$

$$\hat{\lambda}_N = \sum_{n=1}^{N} \frac{Y_n}{N}, \quad \hat{\sigma}_{Y,N}^2 = \sum_{n=1}^{N} \frac{\left(Y_n - \hat{\lambda}_N\right)^2}{N} \tag{10}$$

We define the test statistics as

$$\hat{\delta}_N \equiv N(\hat{m}_N - 1). \tag{11}$$

As $N \to \infty$, testing statistics $\hat{\delta}_N$ has the following asymptotic,

$$\hat{\delta}_N \Rightarrow \delta + \sigma \int_0^1 \sqrt{X_s} dW_s / \int_0^1 X_s ds$$

where X_t is a Cox-Ingersoll-Ross (CIR) process satisfying $dX_t = \sigma\sqrt{X_t}dW_t - \lambda X_t dt$ and W_t is a standard Brownian motion in (13).

Diffusion approximation

Let $\mathscr{F}_n = \sigma(Z_1, \ldots, Z_n, Y_1, \ldots, Y_n)$, we have

$$E(Z_n | \mathscr{F}_{n-1}) = mZ_{n-1} + \lambda$$

Define $\epsilon_n \coloneqq Z_n - E(Z_n | \mathscr{F}_{n-1}) = Z_n - mZ_{n-1} - \lambda$. Then ϵ_n is a martingale difference with respect to $\mathscr{F}_n$, since

$$E(\epsilon_n | \mathscr{F}_{n-1}) = \sum_{k=1}^{Z_{n-1}} E(\xi_{n,k} - m) + E(Y_n - \lambda) = 0,$$

and the conditional variance is

$$E\left(\epsilon_n^2 | \mathscr{F}_{n-1}\right) = \sigma^2 Z_{n-1} + \sigma_Y^2.$$

We can represent Z_n in (4) as

$$Z_n = \lambda + mZ_{n-1} + \epsilon_n \tag{12}$$

which is an integer value autoregressive process (INAR) with martingale differences errors ϵ_n.

From now, instead of N, we set the sample size as $\lfloor\sqrt{c}t\rfloor$, which means the greatest integer less than or equal to $\sqrt{c}t$. And we consider the convergence and the asymptotic properties as $c \to \infty$.

As $c \to \infty$,

$$\frac{1}{\sqrt{c}} \sum_{n=1}^{\lfloor\sqrt{c}t\rfloor} \frac{Z_n - mZ_{n-1}}{\sigma\sqrt{Z_{n-1}}} \Rightarrow W_t \tag{13}$$

where where "$\Rightarrow$" stands for weak convergence and W is a standard Brownian motion. Let $m_c = 1 + \delta/\sqrt{c} \to 1$ and $\sigma_c^2 \to \sigma^2$ as $c \to \infty$. We also let $Z_0/\sqrt{c} \to x_0$, using a telescoping sum and $m = 1 + \delta/\sqrt{c}$, we have

$$\begin{aligned} Z_{\lfloor\sqrt{c}t\rfloor}/\sqrt{c} &= Z_0/\sqrt{c} + \sum_{n=1}^{\lfloor\sqrt{c}t\rfloor} \left(Z_n - Z_{n-1}\right)/\sqrt{c} \\ &= Z_0/\sqrt{c} + \sum_{n=1}^{\lfloor\sqrt{c}t\rfloor} \left(Z_n - Y_n - mZ_{n-1}\right)/\sqrt{c} + \delta \sum_{n=1}^{\lfloor\sqrt{c}t\rfloor} Z_{n-1}/\sqrt{c} + \sum_{n=1}^{\lfloor\sqrt{c}t\rfloor} Y_n/\sqrt{c} \\ &\Rightarrow X_t \equiv x_0 + \sigma \int_0^t \sqrt{X_s} dW_s + \delta \int_0^t X_s ds + \lambda t \end{aligned} \tag{14}$$

where X_t is a Cox-Ingersoll-Ross (CIR) process satisfying $dX_t = \sigma\sqrt{X_t}dW_t - \lambda X_t dt$ with initial value x_0 and W_t is a standard Brownian motion in (13).

3 Asymptotic properties of the sequential testing procedure $(\hat{\delta}_{\tau_c}, \tau_c)$ in SCT

We investigate the asymptotic properties of the sequential testing procedure $(\hat{\delta}_{\tau_c}, \tau_c)$. We approximate the above discrete-time models to the continuous-time models.

For $\delta = 0$, letting $q_t = 4X_t/\sigma^2$, we obtain the squared Bessel process with dimension

$4\lambda/\sigma^2$;

$$q_t = q_0 + 2\int_0^t \sqrt{q_s} dW_s + 4\lambda t/\sigma^2. \tag{15}$$

The hypotheses (5,6) are reduced to in continuous time;

$$\begin{aligned} &\text{Under} \quad P^0 : dX_t = \sigma\sqrt{X_t} dW_t + \lambda dt, \\ &\text{Under} \quad P^\delta : dX_t = \sigma\sqrt{X_t} dW_t + (\delta X_t + \lambda)\, dt. \end{aligned}$$

Using a Girsanov transformation $d\tilde{W}_t = -\delta X_t dt/\sigma + dW_t$, the likelihood process is represented as

$$dP^\delta/dP^0 = \exp\left(\delta \int_0^t \left(\sqrt{X_s}/\sigma\right) dW_s - \delta^2/2 \int_0^t X_s/\sigma^2 ds\right).$$

Then, we have the M.L.E. and observed Fisher information of δ,

$$\tilde{\delta}_t = \delta + \sigma \int_0^t \sqrt{X_s} d\tilde{W}_s / \int_0^t X_s ds, \quad \tilde{I}_t = \int_0^t X_s/\sigma^2 ds,$$

which corresponded to the limit of the $N(\hat{m}_N - 1)$ and the observed Fisher information in discrete time in (8) with $t = 1$.

Define martingale M_t and its quadratic variation $\langle M \rangle_t$ as

$$M_t \equiv \int_0^t \sqrt{X_s/\sigma^2} dW_s, \quad \langle M \rangle_t = \int_0^t X_s/\sigma^2 ds. \tag{16}$$

According to Theorem 7.2 in Ikeda and Watanabe (1989) [3] p.85, letting

$$U_v \equiv \inf\{t \geq 0 : \langle M \rangle_t = v\} = \langle M \rangle_v^{-1}, \tag{17}$$

$\langle M \rangle_{U_v} = v$ and $B_v \equiv M_{U_v}$ becomes a Brownian motion. B_v is so-called a time-changed (or DDS) Brownian motion. Let

$$\rho_v \equiv X_{U_v}/\sigma^2 = d\langle M \rangle / dt|_{t=U_v}, \tag{18}$$

then we can obtain the main theorem as follows.

Theorem 1. *Suppose Z_n is generated by the model (4) with an initial value Z_0 satisfying $Z_0/\sqrt{c} \to x_0$. Then the asymptotic behavior of the stopping times τ_c in (2) and the* *sequential test statistics* $\hat{\delta}_{\tau_c}$ *is given as follows: as* $c \uparrow \infty$,

$$\left(\hat{\delta}_{\tau_c}, \tau_c/\sqrt{c}\right) \Rightarrow \left(\delta + \int_0^{U_1} X_s dW_s, U_1\right) = \left(\delta + B_1, \int_0^1 \rho_s^{-1} ds\right) \tag{19}$$

where B_t *is a standard Brownian motion,* $U_1 \equiv \inf\left\{t : \int_0^t X_s/\sigma^2 ds = 1\right\}$, *and* ρ_t *is the Bessel process with drift* δ, *dimension* $d = 2\lambda/\sigma^2 + 1$, *and initial value* $\rho_0 = x_0/\sigma^2$;

$$d\rho_t = (\frac{\lambda/\sigma^2}{\rho_t} - \delta)dt + dB_t. \tag{20}$$

The joint Laplace transform of (ρ_v, U_v) under H_0 can be obtained from the time change of the squared Bessel process q_t in (15) with $q_0 = 4x_0/\sigma^2 = 4\rho_0$;

$$\int_0^\infty e^{-\gamma v} E_{q_0}^0 \left[\exp\left(-\alpha\rho_v - \beta U_v\right)/\rho_v\right] dv = \int_0^\infty e^{-\beta t} E_{q_0}^0 \left[\exp\left(-\frac{\alpha}{4} q_t - \frac{\gamma}{4}\int_0^t q_s ds\right)\right] dt. \tag{21}$$

Using the Bessel bridge in Pitman and Yor (1982), under H_0 we can obtain,

$$\mathrm{P}_{q_0}\left(\int_0^u \rho_s^{-1} ds \in dt, \rho_u \in dz\right) = \frac{z^{\nu+1}}{q_0^\nu} \mathrm{is}_u\left(2\nu, t/2, 0, q_0 + z, \sqrt{q_0 z}\right) dt dz,$$

with $\nu = d/2 - 1 = \lambda/\sigma^2 - 1/2$. is_u function is special inverse Laplace transform defined as

$$\mathrm{is}_u\left(\nu, t, r, z, x\right) = \mathscr{L}_\gamma^{-1}\left[\left(\frac{\sqrt{2\gamma}}{\sinh\left(t\sqrt{2\gamma}\right)}\right) \exp\left(-r\sqrt{2\gamma} - z\sqrt{2\gamma}\coth\left(t\sqrt{2\gamma}\right)\right) I_\nu\left(\frac{2x\sqrt{2\gamma}}{\sinh(t\sqrt{2\gamma})}\right)\right],$$

where I_ν is the modified Bessel function. See Borodin and Salminen (2002) for the expression of is_u function which includes parabolic cylinder functions. Using Girsanov's theorem, we can obtain the joint probability densities of (ρ_v, U_v) under H_1 with initial value.

The joint Laplace transform can also be obtained as

$$E_{q_0}^0\left[\exp\left(-\alpha\rho_v - \beta U_v\right)/\rho_v\right] = \sum_{n=0}^\infty \sum_{j=0}^\infty \sum_{l=0}^\infty \frac{q_0^n \alpha^j \beta^l}{n!j!l!} \int_0^1 J(t, v, n, j, l) dt, \tag{22}$$

with

$$J(t,v,n,j,l) = \frac{(-1)^n (1-\sqrt{s})^{j+n} (\sqrt{s}+1)^{-j-\nu-n-1} s^{\frac{\nu-3}{4}} \log^l(s)(-n-\nu-1)^{(j)}}{\Gamma\left(\frac{1}{2}(j+l-n+1)\right)}$$
$$\times 2^{\frac{1}{2}(-j-3l-3n-1)+\nu} v^{\frac{1}{2}(j+l-n+1)-1} {}_2F_1\left(-j,-n;-j-n-\nu;\frac{(\sqrt{s}+1)^2}{(\sqrt{s}-1)^2}\right)$$

where $x^{(m)}$ is the factorial power and ${}_2F_1$ is Gauss hypergeometric function. Using Girsanov's theorem, we can obtain the joint Laplace transform of (ρ_v, U_v) under H_1.

4 Sequential probability ratio test

Let $m = 1+\theta$, for a prespecified θ', the following simple hypotheses are considered,

$$H_0 : \theta = 0 \quad \text{vs} \quad H_{\theta'} : \theta = \theta'.$$

Under $H_{\theta'}$, assume $\theta' = \delta'/\sqrt{c}$. Suppose we have a sample (Z_n, Y_n), $n = 1, 2, \ldots, N$ from (4), the likelihood ratio becomes

$$L_N = \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=1}^{N}\left(-2\theta'\left(Z_n - Y_n - Z_{n-1}\right) + \theta'^2 Z_{n-1}\right) + o\left(\frac{1}{N^2}\right)\right\}.$$

The SPRT with boundaries A, B , where $0 < B < A < \infty$, continuous sampling as long as $L_N \in [B, A]$, it stop sampling when either $L_N < B$ or $L_N > A$, and it reject H_0 when $L_N > A$. Let $a = \log A$, $b = \log B$,

$$l_N = \log L_N = \frac{\theta'}{\sigma^2}\sum_{n=1}^{N}(Z_n - Y_n - Z_{n-1}) - \frac{\theta'^2}{2\sigma^2}\sum_{n=1}^{N} Z_{n-1} + o\left(\frac{1}{N^2}\right)$$

define the stopping time as

$$T_c^{(a,b)} = \inf\left\{N > 1 : l_N \notin [b, a]\right\}. \tag{23}$$

Now let α and β be the error probabilities of SPRT. The type 1 error and type II error are

$$\alpha \approx P_{H_0}\left(l_{T_c^{(a,b)}} > a\right) \quad \text{and} \quad \beta \approx P_{H_{\theta'}}\left(l_{T_c^{(a,b)}} < b\right). \tag{24}$$

For prespecified error probabilities α, β, A, B could be determined as follows.

$$A = \frac{1-\beta}{\alpha}, \quad B = \frac{\beta}{1-\alpha}.$$

Let $N = \lfloor \sqrt{c} t \rfloor$, As $c \to \infty$,

$$l_{\lfloor \sqrt{c} t \rfloor} \Rightarrow \delta' \int_0^t \frac{\sqrt{X_s}}{\sigma} dW_s - \frac{\delta'^2}{2} \int_0^t \frac{X_s}{\sigma^2} dW_s = \delta' M_t - \frac{\delta'^2}{2} \langle M \rangle_t$$

where M is the martingale with quadratic variation $\langle M \rangle_t$ defined in (16). Then, since $T_c^{(a,b)} = \inf \left\{ \lfloor \sqrt{c} t \rfloor > 1 : l_{\lfloor \sqrt{c} t \rfloor} \notin [b, a] \right\}$we have,

$$T_c^{(a,b)} / \sqrt{c} \to \inf \left\{ t > 0 : \delta' M_t - \frac{\delta'^2}{2} \langle M \rangle_t \notin [b, a] \right\} \equiv T^{(a,b)}. \tag{25}$$

We approximate the above discrete-time SPRT results to continuous-time models. Let δ' be the prespecified local parameter under alternative. The observed process is X_t is

$$\begin{aligned} H_0 : \quad & X_t = x_0 + \sigma \int_0^t \sqrt{X_s} dW_s + \lambda t, \\ H_{\delta'} : \quad & X_t = x_0 + \sigma \int_0^t \sqrt{X_s} dW_s + \delta' \int_0^t X_s ds + \lambda t. \end{aligned}$$

Let P_{H_0} and $P_{H_{\delta'}}$ be the probability measure under $H_0, H_{\delta'}$, respectively. Define

$$\alpha_t = \exp \left(\int_0^t f_s dW_s - \frac{1}{2} \int_0^t f_s^2 ds \right);$$

and

$$dP_{H_0} = \alpha_t dP_{H_{\delta'}} \text{on} \quad \mathcal{F}_t.$$

According to Girsanov theorem, assume that α_t is a martingale with respect to $\mathcal{F}_t$ and $P_{H_{\delta'}}$. Then the process

$$\tilde{W}_t = W_t + \int_0^t f_s ds$$

is a Brownian motion with respect to $P_{H_{\delta'}}$. Here

$$\alpha_t = \left. \frac{dP_{H_0}}{dP_{H_{\delta'}}} \right|_{\mathcal{F}_t} = \exp \left\{ \delta' M_t - \frac{\delta'^2}{2} \langle M \rangle_t \right\}.$$

Under suitable regularity conditions (Liptser and Shiryaev [13]) , the likelihood ratio is

$$L_t = \exp \left(\delta' M_t - \frac{\delta'^2}{2} \langle M \rangle_t \right).$$

5 Simulation

We conducted simulations to examine the performance of the sequential test for criticality of branching process with immigration. All the simulations reported below were performed using R. We investigate the rejection rates and the operating characteristics (OCs) of the sequential LSE and stopping times.

In Monte Carlo simulation, we let $\xi_n^{(k)} \sim i.i.d.$ Negative Binomial (k, p) and $Y_n \sim i.i.d.$ Poisson (λ) replication= $10,000$ and set initial value $x_0 = 0$, $m = 0.99, 1, 1.01$, $k = 5, \lambda = 10$. It is easy to show $p = k/(m+k)$ and $\sigma^2 = m + m^2/k$.

For data generating process, we set $m = 0.99, 1, 1.01$, which is the "true" parameter. For comparison, we set the prespecified SPRTs with $m_{H_0} = m_0 = 1$ and $m_{H_1} = m'$ where $m' = 0.98, 0.99, 1.01, 1.02$. The simulation results are given in Tables.

Table 1 shows the simulation results of SCT. The value in parentheses is the theoretical value. The reject rates are computed from the standard distribution. Both the reject rates and the sequential maximum likelihood estimation of offspring are close to the theoretical values.

Table 1: Rejection rate (RRs) and OC's of SCT

m	$\sqrt{c}$	RR(%)		τ_c		$\hat{m}_{\tau_c}$	
		Left	Right	Mean	sd	Mean	sd
1	100	5.9	5.6	49.0(49.7)	7.0(5.0)	1.00	0.011(0.01)
	200	5.1	5.5	98.9(99.5)	12.2(10.0)	1.00	0.0051(0.005)
0.99	100	26.4(26.0)		52.7(54.2)	8.15(5.72)	0.99	0.0105(0.01)
	200	64.0(63.9)		116.9(118.8)	15.64(13.04)	0.99	0.0051(0.005)
1.01	100	26.7(26.0)		45.7(45.9)	6.41(4.43)	1.01	0.0106(0.01)
	200	63.7(63.9)		85.7(85.4)	9.72(7.85)	1.01	0.0051(0.005)

Table 2 lists the simulation results of SPRT. We set size $\alpha = 5\%$, power $1-\beta = 63.9\%$. The first column is the "true" m which is data generating process. The second column is the prespecified mean of offspring of $H_1 : m_{H_1} = m'$. The last three columns are the rejection rate, the mean of stopping times and the estimates of m, respectively.

In the SPRT, when we set the prespecified local parameter, the same as the true parameter in date generating process, the simulation results show that the SPRTs with good error probabilities with small expected sample sizes by comparison with SCT with $\sqrt{c} = 200$. However, with the same error probabilities, we set the prespecified parameter m' a little different from the true m, for example, $m' = 0.98$, $m = 0.99$. The simulation shows that the little difference in m' and m makes a big difference in power. Especially, when $m > 1$ (or $m < 1$) in DGP, the prespecified parameter m' is set as $m < 1$ (or $m > 1$), the power of the test becomes almost 0. From the simulations, it is easy to understand that the SPRT is practically inconvenient when true alternative hypothesis

Table 2: Rejection rate (RRs) and OC's of SPRT with $m_{H_0} = 1$, $m_{H_1} = m'$

m	m'	RR(%)	mean $(T_c^{(a,b)})$	$\hat{m}_{T_c^{(a,b)}}$
1	0.98	4.98	41.4	1.00
	0.99	4.75	66.4	1.01
	1.01	4.47	57.9	0.99
	1.02	4.14	36.9	0.99
0.99	0.98	33.15	51.5	0.99
	0.99	69.24	93.0	0.99
	1.01	0.03	41.7	0.98
	1.02	0.30	32.9	0.99
1.01	0.98	0.38	34.5	1.01
	0.99	0.03	42.6	1.01
	1.01	68.02	71.1	1.01
	1.02	31.46	42.7	1.01

in not known.

6 Conclusion

In this paper, we compare two types of sequential test for testing the criticality of the offspring mean m of the branching process with immigration in sequential schemes. SCT employs the stopping time defined by the observed Fisher information of the offspring mean. In SCT, the limiting of the sequential maximum likelihood estimation of offspring mean is asymptotically normally distributed under both null and alternative hypotheses. For the SPRT, the testing result is highly dependent on the prespecified the alternative probability densities. From the practical point of view, the applications of SPRT may lead to a type II error with large probability. In this paper, the simulation results of the OC's of the SPRT are showed. It will be possible to compute the theoretical values of the OC's of SPRT in a similar way as AR(1) model. Research about the computation of the theoretical value is currently under way.

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